today:

midterm: §§ 6.2-6.4, 7.1-7.3

§ 7.4 - partial fractions

tuesday:

homework 5 due (7.3.8, 7.3.22, 7.3.40, 7.4.20, 7.4.48, 7.4.50)

§ 4.4 - l'Hôpital's rule

wednesday:

mslc partial fractions workshop in CH 042 @ 12:30, 1:30, 3:30

thursday, 5 november:

§ 7.8 - improper integrals

mslc l'Hôpital's rule workshop in CH 042 @ 12:30, 3:30

friday, 6 november:

last drop day

webwork 5 due @ 11:55 pm

mslc webwork 5 workshop in SEL 040 @ 12:30, 1:30, 2:30, 3:30, 4:30

simplifying expressions

Recall:

We can combine fractions by getting a common denominator and then adding. For example,

$$\frac{1}{x-1} + \frac{3}{x+2} = \frac{1\,(x+2) + 3\,(x-1)}{(x-1)(x+2)} = \frac{4x-1}{x^2+x-2}$$

simplifying expressions

Since

$$\frac{1}{x-1}+\frac{3}{x+2}=\frac{1\,(x+2)+3\,(x-1)}{(x-1)(x+2)}=\frac{4x-1}{x^2+x-2}$$

it follows that

$$\begin{aligned}\int \frac{4x-1}{x^2+x-2}\mathrm{d}x &= \int \left(\frac{1}{x-1}+\frac{3}{x+2}\right)\mathrm{d}x \\ &= \ln|x-1|+3\ln|x+2|+C\end{aligned}$$

Moral:

If we could turn rational expressions into the difference of fractions, we may make it easier to integrate.

reversing the process

case 0:

That is:

A rational function is a function that can be written as a polynomial divided by a polynomial.

A rational function is a function of the form

$$f(x) = \frac{c_m\, x^m + c_{m-1}\, x^{m-1} + \cdots + c_0}{d_n\, x^n + d_{n-1}\, x^{n-1} + \cdots + d_0}$$

If $m \geq n$ we can long divide to write f as a polynomial plus a rational function where the numerator has smaller degree than the denominator.

example

If the degree of the numerator is bigger than the degree of the denominator, start by doing long division.

By long division,

$$\frac{x^5}{x-1} = x^4 + x^3 + x^2 + x + 1 + \frac{1}{x-1}$$

Thus

$$\int \frac{x^5}{x-1} \mathrm{d}x = \int \left(x^4 + x^3 + x^2 + x + 1 + \frac{1}{x-1} \right) \mathrm{d}x$$
$$= \frac{x^5}{5} + \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x-1| + C$$

After we do long division, we are left with rational functions where the degree of the numerator is smaller than the degree of the denominator.

case i:

Ask:

How would we get this denominator from combining fractions?

Suppose we have a rational function

$$f(x) = \frac{c_m\, x^m + c_{m-1}\, x^{m-1} + \cdots + c_0}{d_n\, x^n + d_{n-1}\, x^{n-1} + \cdots + d_0}$$

with $m < n$ whose denominator can be decomposed into distinct linear factors:

$$f(x) = \frac{c_m\, x^m + c_{m-1}\, x^{m-1} + \cdots + c_0}{(a_1\, x - b_1)\,(a_2\, x - b_2)\cdots(a_n\, x - b_n)}$$

case i:

$$\begin{aligned} f(x) &= \frac{c_m\, x^m + c_{m-1}\, x^{m-1} + \cdots + c_0}{(a_1\, x - b_1)\,(a_2\, x - b_2)\cdots(a_n\, x - b_n)} \\ &= \frac{A_1}{a_1\, x - b_1} + \frac{A_2}{a_2\, x - b_2} + \cdots + \frac{A_n}{a_n\, x - b_n} \end{aligned}$$

We'll talk about how to find the A_i later.

There is nothing special about the names A_1, A_2, ... A_n.... in practice, it's usually easier to call them A, B, C, etc...

example

Find the form of the partial fractions decomposition of

$$\frac{1}{x^3 + 4x^2 - 21x}$$

Answer:

A/x + B/(x-3) + C/(x+7)

case ii:

If only linear factors, but some are repeated.

If the denominator has the factor

$$(a\ x\ \text{-}\ b)^r$$

Use

$$\frac{A_1}{a\,x - b} + \frac{A_2}{(a\,x - b)^2} + \cdots \frac{A_r}{(a\,x - b)^r}$$

example

Find the form of the partial fractions decomposition of

$$\frac{3x+2}{(1+x)^3(1-x)}$$

Answer:

A/(1-x)+B/(1+x)+C/(1+x)^2+D/(1+x)^3

case iii:

Recall: Every polynomial can be factored into a product of linear and quadratic terms.

A non-repeated irreducible quadratic factor.

If the denominator has the factor

$$a\,x^2 + b\,x + c$$

Use

$$\frac{A_1\,x + A_2}{a\,x^2 + b\,x + c}$$

example

Find the form of the partial fractions decomposition of

$$\frac{3x+2}{(1+x)^3(1+x^2)}$$

Answer:

(Ax+B)/(1+x^2)+C/(1+x)+D/(1+x)^2+E/(1+x)^3

case iv:

Recall: Every polynomial can be factored into a product of linear and quadratic terms.

A repeated irreducible quadratic factor.

If the denominator has the factor

$$(a\ x^2+b\ x+c)^r$$

Use

$$\frac{A_1\,x+B_1}{a\,x^2+b\,x+c}+\frac{A_2\,x+B_2}{(a\,x^2+b\,x+c)^2}+\cdots\frac{A_r\,x+B_r}{(a\,x^2+b\,x+c)^r}$$

example

Find the form of the partial fractions decomposition of

$$\frac{1-x}{(1+x)(1+x^2)^3}$$

Answer:

A/(1+x)+(Bx+C)/(1+x^2)+(Dx+E)/(1+x^2)^2 +(Fx+G)/(1+x^2)^3

how to find the constants

option 1

- cross multiply
- denominators become the same (so ignore)
- match coefficients on $1, x, x^2$, etc...

example

Find the partial fractions decomposition of

$$\frac{1}{x^3 + 4x^2 - 21x}$$

Found the form earlier.

Answer:

1 / (70(x+7)) + 1 / (30(x-3)) - 1 / (21x)

option II

- cross multiply
- denominators become the same (so ignore)
- plug in various values for x to get a system of equations, solve

example

Find the partial fractions decomposition of

$$\frac{7x-1}{(1+x)(1-x)}$$

Good choices to plug in are 1 and -1.

Answer:

-4/(x+1) - 3/(x-1)

example

evaluate

$$\int \frac{5x+1}{(x^2+1)(x+1)}\,\mathrm{d}x$$

Answer:

ln | x^2+1 | -2 ln | x+1 | + 3 arctan(x) + C

Sometimes we have to do other techniques first to get the integral in a form where we can use partial fractions. Here, we first make a u-substitution.

example

evaluate

$$\int \frac{\mathrm{d}x}{x\sqrt{x+1}}$$

This is Stewart 7.4.39.

Evaluate by first letting u=sqrt(x+1)

(Then dx = 2u du)

Answer:

2 ln | sqrt(x+1) - 1 | - ln | x | + C

coming soon

- read § 4.4
- homework 5 due next tuesday
- start webwork 5, due next friday
- start extra credit project 2,
  due 16 november @ 6:00 am